

COPLANAR k -UNDULOIDS ARE NONDEGENERATE

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ABSTRACT. We prove each embedded, constant mean curvature (CMC) surface in Euclidean space with genus zero and finitely many coplanar ends is nondegenerate: there is no nontrivial square-integrable solution to the Jacobi equation, the linearization of the CMC condition. This implies that the moduli space of such coplanar surfaces is a real-analytic manifold and that a neighborhood of these in the full CMC moduli space is itself a manifold. Nondegeneracy further implies (infinitesimal and local) rigidity in the sense that the asymptotes map is an analytic immersion on these spaces, and also that the coplanar classifying map is an analytic diffeomorphism.

1. INTRODUCTION

The Delaunay unduloids [Del] are surfaces of revolution with constant mean curvature $H \equiv 1$; they are singly-periodic surfaces interpolating between a cylinder of diameter 1 and a chain of unit spheres. If a finite topology surface M is properly embedded in \mathbb{R}^3 with constant mean curvature (CMC), Korevaar, Kusner and Solomon [KKS] proved that each end of M is asymptotic to an unduloid, and if M is two-ended then it must be an unduloid. Their results motivate the following global rigidity question: Do these unduloid asymptotes determine M uniquely?

To rephrase this question more formally, we define the *asymptotes map* $\mathcal{A} : \widehat{\mathcal{M}}_{g,k} \rightarrow \mathcal{U}^k$ assigning to any CMC surface M its k unduloid asymptotes. Here $\widehat{\mathcal{M}}_{g,k}$ denotes the premoduli space of all CMC surfaces with genus g and k ends, and $\mathcal{U} := \widehat{\mathcal{M}}_{0,2}$ is the space of unduloids. (A precise definition—including a weakened notion of embeddedness due to Alexandrov [Ale]—is given in Section 2.1.) In general, $\widehat{\mathcal{M}}_{g,k}$ is a real-analytic variety [KMP] and \mathcal{A} is a real-analytic map [Kus]. Thus a strong form of the above question becomes: Is the asymptotes map \mathcal{A} an embedding?

While it appears difficult to answer this question fully, it follows from [KMP, Kus] that \mathcal{A} is an embedding in a neighborhood of any CMC surface M which is *nondegenerate* in the sense that $u \equiv 0$ is the only L^2 solution to the Jacobi equation

$$(1.1) \quad \mathcal{L}(u) := \Delta u + |A|^2 u = 0.$$

Here Δ is the Laplace-Beltrami operator and $|A|^2$ is the squared length of the second fundamental form of M . The Jacobi operator \mathcal{L} is the linearization at M of the mean curvature operator, and so (by the implicit function theorem) near a nondegenerate surface M , the premoduli space $\widehat{\mathcal{M}}_{g,k}$ is a $3k$ -dimensional real-analytic manifold [KMP]. (It follows that the moduli space $\mathcal{M}_{g,k}$ —where surfaces differing by a rigid motion of \mathbb{R}^3 are identified—has dimension $3k - 6$ near M .)

At a nondegenerate M , we thus get local rigidity, in the sense that no nearby surfaces have the same asymptotes, and also infinitesimal rigidity, in the sense that every nonzero Jacobi field on M has nonzero first-order effect on the asymptotes (Proposition 4.6).

In this paper we consider *coplanar* CMC surfaces, whose asymptotic axes all lie in a common plane. Any coplanar CMC surface has mirror symmetry across that plane [KKS]. We will call a coplanar CMC surface with genus zero and k ends a *coplanar k -unduloid*, and we let $\mathcal{M}'_k \subset \mathcal{M}_{0,k}$ denote the moduli space of all coplanar k -unduloids.

No classification results are yet available for CMC surfaces with higher genus or noncoplanar ends, but we do know [GKS1, GKS2] that \mathcal{M}'_k is homeomorphic to a certain connected $(2k-3)$ -manifold \mathcal{D}_k of spherical metrics on the disk.

Our principal result here is the following:

Nondegeneracy Theorem. *All coplanar k -unduloids are nondegenerate.*

Nondegeneracy implies that, in a neighborhood of any coplanar k -unduloid, $\mathcal{M}_{0,k}$ is a real-analytic $(3k-6)$ -manifold and furthermore that \mathcal{M}'_k is an analytic $(2k-3)$ -submanifold. In other words, $\mathcal{M}'_k \subset \mathcal{M}_{0,k}$ has a tubular neighborhood of dimension $3k-6$, in which \mathcal{M}'_k is analytically embedded as a submanifold of codimension $k-3$.

Another common application of nondegeneracy is the construction of new CMC surfaces from old. Our result gives a wide class of surfaces to which one can apply gluing constructions [MP, MPP, Rat], which typically require the summands to be nondegenerate.

There is no example of an Alexandrov-embedded CMC surface known to be degenerate (other than \mathbb{S}^2). But without this embeddedness assumption, there are many examples of degenerate CMC surfaces, including of course all compact CMC surfaces as well as unduloids with bubbletons.

Essential to the proof of our theorem is an understanding of the formal tangent space at M to the premoduli space $\widehat{\mathcal{M}}_{g,k}$, which is the space $\widehat{\mathcal{J}}(M)$ of all tempered Jacobi fields (see Section 2.5). If M is coplanar, the reflection symmetry allows us to decompose any Jacobi field into even and odd parts, preserved and reversed (respectively) by the symmetry. The maximum principle can be used to show [KKR] that, up to equivalence, the only bounded odd Jacobi field is the one arising from translation perpendicular to the mirror plane; this is not in L^2 . Thus all L^2 fields are even. We let $\widehat{\mathcal{J}}'(M)$ be the space of even, tempered Jacobi fields, and $\widehat{\mathcal{J}}'_0(M)$ be the subspace of L^2 fields. The first can be viewed as the subspace tangent at M to the corresponding premoduli space of coplanar CMC surfaces.

Our proof of the nondegeneracy theorem relies on the construction of a formal differential $\partial\Phi$ for the classifying map $\Phi: \mathcal{M}'_k \rightarrow \mathcal{D}_k$ of [GKS2]. We prove that $\partial\Phi$ is injective in order to bound the dimensions of $\widehat{\mathcal{J}}'(M)$ and $\widehat{\mathcal{J}}'_0(M)$. This injectivity also leads to our second main result:

Diffeomorphism Theorem. *The classifying map $\Phi: \mathcal{M}'_k \rightarrow \mathcal{D}_k$ for coplanar k -unduloids is a real-analytic diffeomorphism.*

To understand the differential $\partial\Phi$, let us recall the construction of Φ : a coplanar k -unduloid M is decomposed by its symmetry plane into two halves M^\pm , each a disk. The CMC disk M^+ has a conjugate cousin \widetilde{M}^+ , a minimal disk in \mathbb{S}^3 isometric to M^+ , whose boundary lies along k Hopf circles. Hopf projection immerses this disk into \mathbb{S}^2 , giving the spherical metric $\Phi(M)$ with exactly k completion boundary points.

The first step in defining $\partial\Phi$ is thus a linearization of the conjugate cousin construction. This was defined in [KKR], and we reinterpret it in Section 3.1: Given any Jacobi field on M^+ , we get a cousin field on \tilde{M}^+ ; this can be transplanted back to M^+ via the isometry.

Reminiscent of the interchange of Dirichlet and Neumann boundary data for conjugate harmonic functions, one might hope that even fields would conjugate to odd fields and vice versa. Unfortunately, this is not always true, but the conjugate of an even field satisfies a slightly weaker condition—we say it is *almost odd*. We also define the corresponding notion of *almost even* fields, and show that a field is almost even if and only if its conjugate is almost odd.

To complete the definition of $\partial\Phi$, as a linear map on $\hat{\mathcal{J}}'(M)$ modulo Killing fields, we note (Lemma 3.6) that almost odd fields are exactly those which have a well-defined first-order effect on the Hopf projection of the boundary of \tilde{M}^+ .

One main tool is Theorem 5.1, which says the map $\partial\Phi$ is injective, implying the upper bound $\dim \hat{\mathcal{J}}'(M) \leq 2k$. Key to this proof is a differential version (Lemma 4.4) of the fact that the asymptotic necksizes of a coplanar k -unduloid M are readily visible in both $\mathcal{A}(M)$ and $\Phi(M)$.

Finally, the relative index computation of [KMP] can be adapted to the equivariant setting; this shows $\dim \hat{\mathcal{J}}'(M) \geq 2k$, with equality if and only if M is nondegenerate. Our upper bound shows we do have equality, completing the proof of the Nondegeneracy Theorem. Nondegeneracy, in turn, implies that \mathcal{M}'_k is a smooth manifold whose tangent space is the domain of $\partial\Phi$; thus the injectivity of $\partial\Phi$ also gives the Diffeomorphism Theorem.

We expect that our methods might extend to cover the space $\mathcal{M}'_{g,k}$ of coplanar surfaces of higher genus. New methods, however, not based on the mirror symmetry of a coplanar surface, would be needed to answer the question of whether all Alexandrov-embedded CMC surfaces $M \in \mathcal{M}_{g,k}$ are nondegenerate.

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2. BACKGROUND

2.1. Alexandrov-embedded CMC surfaces of finite topology. We study constant mean curvature (CMC) surfaces of finite topology. Here, CMC surfaces are always scaled and oriented (with inward unit normal ν) to have $H \equiv 1$, and a surface of finite topology is $\Sigma := \bar{\Sigma} \setminus \{p_1, \dots, p_k\}$, where $\bar{\Sigma}$ is the closed surface of genus g and each p_i corresponds to an end E_i of Σ .

We are interested in embedded surfaces, but unlike the case of minimal surfaces, where the maximum principle guarantees that embeddedness is preserved under deformations, for CMC surfaces it turns out to be much more natural to consider a slight weakening of the embeddedness condition. A proper CMC immersion $f: \Sigma \rightarrow \mathbb{R}^3$ of finite topology is *Alexandrov-embedded* if each end is embedded and if f extends to a proper immersion $f: W \rightarrow \mathbb{R}^3$. Here W is a three-manifold (necessarily a handlebody of genus g) bounded by Σ , and the inward normal ν points into W along Σ .

Definition. Fixing a topological surface Σ of genus g with k labeled ends E_1, \dots, E_k , we let $\hat{\mathcal{M}}_{g,k}$ denote the *premoduli space* of complete, Alexandrov-embedded CMC immersions $f: \Sigma \rightarrow \mathbb{R}^3$,

modulo reparametrizations of the domain that preserve the labeling. The quotient of $\widehat{\mathcal{M}}_{g,k}$ by the action of Euclidean motions is the *moduli space* $\mathcal{M}_{g,k} := \widehat{\mathcal{M}}_{g,k}/E(3)$ of such CMC surfaces of genus g with k ends. We equip these spaces with the topology given by Hausdorff distance on compact sets.

Delaunay [Del] classified the CMC surfaces of revolution; the embedded ones, called *unduloids*, are periodic and are determined up to rigid motion by their necksize $n \in (0, \pi]$, the length of the shortest closed geodesic. The case $n = \pi$ is the cylinder (of radius $1/2$, to give $H = 1$), while the limit $n \rightarrow 0$ approaches a singular chain of unit spheres.

The only Alexandrov-embedded surfaces for $k \leq 2$ are the sphere and the unduloids [Ale, Mee, KKS]. That is, $\mathcal{M}_{g,k}$ is empty for $g > 0$ when $k \leq 2$, while $\mathcal{M}_{0,0} \cong \{*\}$, $\mathcal{M}_{0,1} = \emptyset$ and $\mathcal{M}_{0,2} \cong (0, \pi]$. By contrast, for $k \geq 3$, Kapouleas [Kap] showed that all $\mathcal{M}_{g,k}$ are nonempty. Our results here establish that an open set in $\mathcal{M}_{0,k}$ has the “expected” dimension $3k - 6$.

We let $\mathcal{U} := \widehat{\mathcal{M}}_{0,2}$ denote the premoduli space of unduloids, and note that an unduloid $U \in \mathcal{U}$ is specified by the following data: its *axis* (an oriented line in \mathbb{R}^3), its *necksize* $n \in (0, \pi]$, and (when $n \neq \pi$) its *phase* (the location of the necks along the axis). The necksize and phase together give a point in an open 2-disk, so \mathcal{U} can be viewed as a disk-bundle (indeed, a product) over the space of oriented lines in \mathbb{R}^3 . (Compare [Kus].)

The unduloids are important for us because, as we stated in Section 1, each end of an Alexandrov-embedded CMC surface is asymptotic to some unduloid [KKS]. This motivates us to define the asymptotes map \mathcal{A} , and to view our Nondegeneracy Theorem as an infinitesimal rigidity statement.

2.2. Coplanar k -unduloids. Our arguments apply to surfaces with a certain symmetry. Alexandrov’s reflection technique [Ale] can be adapted [KKS] to show that if $M \in \mathcal{M}_{g,k}$ lies in a half-space bounded by a plane \bar{P} , then it has mirror symmetry across a plane P parallel to \bar{P} . Furthermore, P cuts M into two mirror halves M^\pm , each of which is a graph over a region immersed in P . We conclude that the asymptotic axes of such a surface all lie in P , and call such a surface *coplanar*.

We agree to normalize a coplanar surface M such that P is the horizontal \mathbf{ij} -plane. We denote by $\widehat{\mathcal{M}}'_{g,k} \subset \widehat{\mathcal{M}}_{g,k}$ and $\mathcal{M}'_{g,k} \subset \mathcal{M}_{g,k}$ the subspaces consisting of coplanar surfaces with this symmetry plane. The *upper half* of $M \in \mathcal{M}'_{g,k}$ is $M^+ := \{p \in M : \langle p, \mathbf{k} \rangle \geq 0\}$.

Because we work with surfaces whose ends are labeled, the moduli spaces of coplanar surfaces are disconnected. In this paper, we deal mainly with the case $g = 0$, meaning that M is a k -punctured sphere and M^+ is a closed disk with k boundary points removed. (Note that f is a proper immersion on M^+ , whose k ends correspond to those of M .) The boundary $\partial M^+ = M \cap P$ consists of k oriented curves γ_i , each connecting a pair of ends. The space $\mathcal{M}'_{0,k}$ has $(k-1)!$ components, corresponding to the different cyclic orderings of the ends. We focus attention on one component by assuming the ends of M are labeled cyclically, in the sense that γ_i runs from E_i to E_{i+1} .

Definition. A coplanar CMC surface of genus 0, with $k \geq 3$ ends labeled in cyclic order, is called a *coplanar k -unduloid*. We denote by $\widehat{\mathcal{M}}_k \subset \widehat{\mathcal{M}}'_{0,k}$ and $\mathcal{M}'_k \subset \mathcal{M}'_{0,k}$ the subspaces consisting of coplanar k -unduloids.

The classification result of [GKS2] shows that \mathcal{M}'_k is a connected $(2k-3)$ -manifold, showing that $(k-1)!$ is indeed the number of connected components of $\mathcal{M}'_{0,k}$.

When we write a coplanar k -unduloid $M \in \mathcal{M}'_k$ as $M = f(\Sigma) \subset \mathbb{R}^3$, this is a slight abuse of notation: First, when considering variations, we are often interested in the particular parametrization f , rather than the equivalence class under diffeomorphisms of the domain. Second, although a nonembedded surface M cannot be identified with its image $f(\Sigma) \subset \mathbb{R}^3$, we usually ignore this distinction, and think of points on M as being simply points in \mathbb{R}^3 . Finally, we often ignore the distinction between M and the abstract surface Σ altogether.

2.3. Geometry of the three-sphere. Our arguments are based on the construction of a conjugate cousin, a minimal surface in the three-sphere. We identify \mathbb{S}^3 with the unit quaternions in $\mathbb{H} = \mathbb{R}[1, \mathbf{i}, \mathbf{j}, \mathbf{k}] \cong \mathbb{R}^4$, and $\mathbb{R}^3 = T_1\mathbb{S}^3$ with the (pure) imaginary quaternions. As in any Lie group, we can use left-translation to identify the tangent spaces to \mathbb{S}^3 at any two points. In particular, a vector $\mathbf{u} \in T_1\mathbb{S}^3 \cong \mathbb{R}^3$ is associated to $p\mathbf{u} \in T_p\mathbb{S}^3$, given by quaternion multiplication.

Important for us will be the *Hopf projection* from \mathbb{S}^3 to \mathbb{S}^2 . Given $\mathbf{u} \in \mathbb{S}^2$, a unit imaginary quaternion, the \mathbf{u} -Hopf projection is defined by $\Pi_{\mathbf{u}}(p) := p\mathbf{u}p^{-1}$. The preimage of a point under $\Pi_{\mathbf{u}}$ is a great circle in \mathbb{S}^3 , called a *\mathbf{u} -Hopf circle*. The \mathbf{u} -Hopf circle through $p \in \mathbb{S}^3$ has tangent vector $p\mathbf{u}$ there; in other words, the tangent vectors to the \mathbf{u} -Hopf circles form the left-invariant vector field with value \mathbf{u} at $1 \in \mathbb{S}^3$. The collection of all \mathbf{u} -Hopf circles, for fixed \mathbf{u} , foliates \mathbb{S}^3 and is called the \mathbf{u} -Hopf fibration. Usually, we take $\mathbf{u} = \mathbf{k}$; we call \mathbf{k} -Hopf circles simply Hopf circles.

We will be interested in the infinitesimal isometries (Killing fields) on \mathbb{R}^3 and \mathbb{S}^3 , because they restrict to Jacobi fields on CMC surfaces and their conjugates. The Killing fields on \mathbb{R}^3 are generated by infinitesimal translations and rotations. Given a vector $\mathbf{u} \in \mathbb{R}^3$ the translation by \mathbf{u} is the constant vector field $\tau_{\mathbf{u}}(p) = \mathbf{u}$, while the rotation around \mathbf{u} is $\rho_{\mathbf{u}}(p) = p\mathbf{u} - \mathbf{u}p = -2\mathbf{u} \times p$. (We include the factor 2 because we think of a finite rotation quaternionically as $p \mapsto e^{-t\mathbf{u}}pe^{t\mathbf{u}}$, which rotates by angle $2t$ around the axis \mathbf{u} .) Similarly, the Killing fields on \mathbb{S}^3 are generated by infinitesimal left-translations and right-translations. Given $\mathbf{u} \in \mathbb{R}^3 = T_1\mathbb{S}^3$, the left-translation by \mathbf{u} is the (right-invariant) field $\ell_{\mathbf{u}}(p) = \mathbf{u}p$, while the right-translation by \mathbf{u} is the (left-invariant) field $r_{\mathbf{u}}(p) = p\mathbf{u}$. Note that $r_{\mathbf{u}}$ is the field of unit tangent vectors to the \mathbf{u} -Hopf circles.

2.4. Conjugate cousins and the classifying map. Any simply connected CMC surface in \mathbb{R}^3 has a conjugate cousin minimal surface in \mathbb{S}^3 . We are interested in the case of the upper half M^+ of a coplanar k -unduloid $M \in \mathcal{M}'_k$.

We pull back the metric on $M \subset \mathbb{R}^3$ to Σ , meaning that f is by definition an isometry. (Indeed, we could identify Σ with M .) We let J denote a rotation by $\pi/2$ in the tangent plane to M (or equivalently, Σ), with sign chosen so that $\mathbf{u} \times J\mathbf{u}$ is the inward normal ν to M .

Then the conjugate cousin of M^+ is a minimal surface $\tilde{M}^+ = \tilde{f}(\Sigma^+) \subset \mathbb{S}^3$. Here $\Sigma^+ := f^{-1}(M^+)$ and the immersions f and \tilde{f} are related by the first-order cousin equation

$$(2.1) \quad d\tilde{f} = \tilde{f} df \circ J$$

of [GKS1]. From this equation, we see that \tilde{f} is also an isometry, and that the normal ν of M^+ left translates to the normal $\tilde{\nu} = \tilde{f}\nu$ of \tilde{M}^+ .

As an example, if M is the unit sphere $\mathbb{S}^2 = \mathbb{R}^3 \cap \mathbb{S}^3$, then $\tilde{f} = f$. (This of course depends on—and indeed characterizes—our sign convention for J .)

Lawson's original second-order description [Law] defined conjugate cousins only up to rigid motion. But in our first-order approach, given f , the conjugate \tilde{f} is determined up to a left-translation in \mathbb{S}^3 . Conversely, given a minimal $\tilde{f}: \Sigma^+ \rightarrow \mathbb{S}^3$, equation (2.1) determines f up to translation in \mathbb{R}^3 .

Each of the k boundary curves γ_i of M^+ has constant conormal \mathbf{k} , since it lies in a horizontal plane. It follows from (2.1) that the cousin boundary curves $\tilde{\gamma}_i$ are Hopf circles. Thus under \mathbf{k} -Hopf projection, $\tilde{\gamma}_i$ collapses to a point $\mathbf{p}_i \in \mathbb{S}^2$.

On the other hand, the fact that M^+ is a graph means that its normal points downwards: $\langle \nu, \mathbf{k} \rangle \leq 0$, and the strong maximum principle implies the inequality is strict on the interior. There we have $\langle \tilde{\nu}, \tilde{f}\mathbf{k} \rangle = \langle \nu, \mathbf{k} \rangle < 0$, which means the interior of the surface \tilde{M}^+ is transverse to Hopf circles.

It follows that the map $\Pi_{\mathbf{k}} \circ \tilde{f}: \Sigma^+ \rightarrow \mathbb{S}^2$ is an immersion on the interior [GKS2], so it induces a spherical metric on the open disk. The completion boundary of this metric consists exactly of k points, corresponding to the $\tilde{\gamma}_i$ and developing to the \mathbf{p}_i . We denote by \mathcal{D}_k the space of such *k-point metrics*: spherical metrics on the open disk with k completion boundary points.

This discussion has defined a real-analytic map $\Phi: \mathcal{M}'_k \rightarrow \mathcal{D}_k$, taking M to the metric induced by $\Pi_{\mathbf{k}} \circ \tilde{f}$. The main result of [GKS2] is that Φ is a homeomorphism, explaining why we call it the *classifying map*.

The space \mathcal{D}_k projects onto a certain space \mathcal{T}_k of k -tuples in \mathbb{S}^2 modulo rotation, by mapping a metric $D \in \mathcal{D}_k$ to the images \mathbf{p}_i of its k completion points (under a developing map to the sphere). The space \mathcal{T}_k consists of k -tuples where consecutive points (in cyclic order) are distinct, and furthermore omits k -tuples of the form (p, q, \dots, p, q) which use only two points of \mathbb{S}^2 . Since rotations act freely on such k -tuples, \mathcal{T}_k is a $(2k - 3)$ -manifold. The projection $\mathcal{D}_k \rightarrow \mathcal{T}_k$ is a local diffeomorphism [GKS2]; a k -point metric is determined by its completion points together with some extra combinatorial data. For local considerations, it therefore suffices to consider just the boundary points \mathbf{p}_i instead of the spherical metric $\Phi(M)$. In particular, when we study the differential of the classifying map, we will make use of the fact that

$$(2.2) \quad T_{\Phi(M)} \mathcal{D}_k \cong (T_{\mathbf{p}_1} \mathbb{S}^2 \times \dots \times T_{\mathbf{p}_k} \mathbb{S}^2) / \mathfrak{so}_3.$$

(Here, $\mathfrak{so}_3 = \{\rho_{\mathbf{u}}\}$ is the space of infinitesimal rotations of $\mathbb{S}^2 \subset \mathbb{R}^3$.)

Important later will be the fact [GKS1] that the spherical distance between successive points \mathbf{p}_{i-1} and \mathbf{p}_i is the necksize n_i of the end E_i . It is easy to verify this for an unduloid and its classifying 2-point metric, by explicitly computing the cousin surface, a spherical helicoid. The general case then follows from the fact that E_i is exponentially asymptotic to an unduloid.

2.5. Jacobi fields and nondegeneracy. To understand the local structure of CMC moduli spaces, we study the linearized problem. Suppose $M = f(S)$ is a CMC surface, where we write the domain as S to emphasize that we might be talking about a k -unduloid or about just its upper half. If we vary f in a one-parameter family of immersions $f^t: S \rightarrow \mathbb{R}^3$ with $f^0 = f$, the *variation field* $\dot{f} := \frac{d}{dt} \big|_{t=0} f^t$ of this family can be thought of as an \mathbb{R}^3 -valued vector field on M . (In general, given a one-parameter family f^t , we will use a dot to denote the first derivative with respect to the variation parameter t , evaluated at $t = 0$.)

Definition. An \mathbb{R}^3 -valued vector field V on M is a *Jacobi field* if, for every one-parameter family $f^t: S \rightarrow \mathbb{R}^3$ with variation field $\dot{f} = V$, the mean curvature H^t is constant to first order: $\dot{H} \equiv 0$. We call a Jacobi field *integrable* if it is the first variation of a one-parameter family of CMC surfaces.

Since tangential fields serve merely to reparametrize the surface, this notion clearly depends only on the normal part $V^\perp := \langle V, \nu \rangle$. Indeed, V is a Jacobi field if and only if V^\perp is a *scalar Jacobi field*, satisfying the Jacobi equation (1.1).

Definition. We call a CMC surface M *nondegenerate* if the only L^2 Jacobi fields on M are tangential, or in other words, if the only solution $u \in L^2$ to the Jacobi equation (1.1) is $u \equiv 0$.

The appropriate space of Jacobi fields to study the local structure of the moduli space $\mathcal{M}_{g,k}$, in particular to apply the implicit function theorem, consists of fields more general than L^2 :

Definition. We call a Jacobi field *tempered* if it has sub-exponential growth on each end.

We note that by the asymptotics result of [KKS] every integrable Jacobi field is tempered.

In the premoduli space $\widehat{\mathcal{M}}_{g,k}$, surfaces differing merely by reparametrization are identified; this means that tangential Jacobi fields have no effect on the element $M \in \widehat{\mathcal{M}}_{g,k}$. This leads us to define $\widehat{\mathcal{J}}(M)$ as the quotient of the space of all tempered Jacobi fields modulo tangential fields. Clearly this is isomorphic to the space of normal (or scalar) tempered Jacobi fields, but often it is more convenient to work with equivalence classes, rather than having to take normal parts.

In the moduli space $\mathcal{M}_{g,k}$, surfaces are further identified if they differ by rigid motion; Killing fields have no effect on the element $M \in \mathcal{M}_{g,k}$. Thus we define $\mathcal{J}(M)$ as the quotient of $\widehat{\mathcal{J}}(M)$ modulo Killing fields. For a Jacobi field V , we write $[V]$ for its coset in $\mathcal{J}(M)$.

2.6. Even and odd fields. Suppose V is a smooth \mathbb{R}^3 -valued vector field (for instance, a Jacobi field) along a coplanar k -unduloid M . Writing $\sigma : (x, y, z) \mapsto (x, y, -z)$ for the reflection across the symmetry ij -plane, we can decompose the vector field V into even and odd parts: $V = V_+ + V_-$, where

$$V_\pm(p) := \frac{1}{2}(V(p) \pm \sigma V(\sigma p)).$$

Now consider the behavior along the symmetry curves γ_i of these even and odd parts V_\pm and of their conormal derivatives $\partial_n V_\pm$. We find that V_+ and $\partial_n V_-$ are horizontal (perpendicular to \mathbf{k}), while V_- and $\partial_n V_+$ are vertical (parallel to \mathbf{k}).

We will often consider vector fields V defined only on the upper half M^+ . The discussion above motivates the following definitions.

Definition. Given a vector field V on M^+ , consider its behavior on one of the boundary curves γ_i . We say that V is *even* along γ_i if V is horizontal and $\partial_n V$ is vertical there. Similarly, V is *odd* along γ_i if V is vertical and $\partial_n V$ is horizontal. If one of these conditions holds along all k boundary curves, we simply say V is even or odd, respectively.

Note that if V is odd along γ_i , then its normal component $V^\perp = \langle V, \nu \rangle$ vanishes (i.e., has Dirichlet boundary data) along γ_i . Similarly, if V is even along γ_i then V^\perp has Neumann boundary data in

the sense that $\partial_n V^\perp = 0$ along γ_i . Conversely, if a function u is Dirichlet (or Neumann), then the normal field $V = u\nu$ is odd (or even, respectively).

Now consider a Jacobi field on M^+ . Typically, it cannot be extended to a Jacobi field on all of M . But an even or odd Jacobi field on M^+ does extend, by even or odd reflection, respectively.

Starting with a coplanar surface M , let $\widehat{\mathcal{J}}'(M)$ denote the space of even tempered Jacobi fields (modulo tangent fields). If we vary M within $\widehat{\mathcal{M}}'_{g,k}$, the symmetry plane is fixed, and the first variation V is even and tempered. That is, we can view $V \in \widehat{\mathcal{J}}'(M)$. Indeed, within $\widehat{\mathcal{J}}'(M)$, the cone of integrable Jacobi fields is the tangent cone to $\widehat{\mathcal{M}}'_{g,k}$. Similarly, $\mathcal{J}'(M) \subset \mathcal{J}(M)$ denotes the subspace consisting of cosets of even fields (modulo Killing fields).

Remark. Results of [KMP] show that $\mathcal{M}_{g,k}$ is locally a real-analytic variety; these were adapted in [GKS2] to show the same is true of $\mathcal{M}'_{g,k}$. As varieties, these spaces have formal tangent spaces at every point M . It is straightforward to check that these formal tangent spaces are (isomorphic to) $\mathcal{J}(M)$ and $\mathcal{J}'(M)$, respectively.

The tangent cone $T_M \mathcal{M}_{g,k}$ to $\mathcal{M}_{g,k}$ at M is the cone in $\mathcal{J}(M)$ consisting of all (cosets of) integrable Jacobi fields. Similarly

$$T_M \mathcal{M}'_{g,k} = T_M \mathcal{M}_{g,k} \cap \mathcal{J}'(M)$$

consists of the integrable even Jacobi fields.

3. COUSIN JACOBI FIELDS AND THE DIFFERENTIAL OF THE CLASSIFYING MAP

Our first main technical tool is the conjugation of Jacobi fields introduced in [KKR]. We redevelop this theory with two changes: we use vector-valued Jacobi fields and we work directly in \mathbb{S}^3 . Then we employ this to compute the (formal) differential of the classifying map $\Phi: \mathcal{M}'_k \rightarrow \mathcal{D}_k$.

Throughout this section, we fix a coplanar k -unduloid $M \in \mathcal{M}'_k$, and consider the conjugate \widetilde{M}^+ of its upper half M^+ . Thus \widetilde{M}^+ is a minimal surface in \mathbb{S}^3 .

3.1. Conjugation of Jacobi fields. A Jacobi field for the minimal surface \widetilde{M}^+ means a variation vector field which preserves the minimality condition $H \equiv 0$ to first order. We think of such a field as a map $W: \Sigma^+ \rightarrow \mathbb{H}$ whose value at $x \in \Sigma^+$ is tangent to \mathbb{S}^3 (though not necessarily to \widetilde{M}^+) at $\widetilde{f}(x)$.

Now suppose V is an integrable Jacobi field on M^+ , the initial velocity of a one-parameter family f^t of CMC immersions. At each t , there is a cousin minimal immersion $\widetilde{f}^t: \Sigma^+ \rightarrow \mathbb{S}^3$, well-defined up to left-translation. This one-parameter family has initial velocity $\widetilde{V} := \frac{d}{dt}|_{t=0} \widetilde{f}^t$, a Jacobi field on \widetilde{M}^+ . We call \widetilde{V} a *cousin* of V ; it is well-defined up to an infinitesimal left-translation ℓ_u .

Differentiating the cousin equation $d\widetilde{f}^t = \widetilde{f}^t df^t \circ J^t$ at $t = 0$ gives the first-order linear system

$$(3.1) \quad d\widetilde{V} = \widetilde{V} df \circ J + \widetilde{f} dV \circ J + \widetilde{f} df \circ J$$

relating \widetilde{V} and V . We will discuss the meaning of $J := \frac{d}{dt}|_{t=0} J^t$ below.

We have derived (3.1) assuming that V is integrable. But in fact, we can start with *any* Jacobi field V and solve (3.1) to give a cousin \widetilde{V} . This was shown in [KKR] for the special case when V is normal. The general case then follows by linearity since any tangential field is integrable.

Alternatively, we can simply repeat the argument of [KKR] for the general case: Any point $x \in \Sigma^+$ has a neighborhood N on which the initial surface $f|_N$ is stable for the CMC variational problem of minimizing area with fixed enclosed volume. On such a stable CMC disk, a standard implicit function argument shows that all Jacobi fields are integrable [Whi]. Thus (3.1) holds on the neighborhood N of the arbitrary point x , meaning that it holds everywhere.

Multiplying (3.1) on the left by \tilde{f}^{-1} and composing on the right with J , we can solve for dV :

$$(3.2) \quad dV = -\tilde{f}^{-1}(d\tilde{V} \circ J + \tilde{V} df) + df \circ \dot{J} \circ J.$$

Note that in both equations, (3.1) and (3.2), we have left the \dot{J} term implicit. It can be computed equally well from V or from \tilde{V} . To see this, we again use the fact that Jacobi fields are locally integrable to express \dot{J} as the rate of change of conformal structure in a one-parameter family f^t , or equivalently in the isometric family \tilde{f}^t .

We have remarked that V determines \tilde{V} up to a left-translation $\ell_{\mathbf{u}}$; similarly \tilde{V} determines V up to a translation $\tau_{\mathbf{u}}$. These facts are also easy to check from the equations we have derived. Using linearity, we reduce to the case that the starting field vanishes, in which case $\dot{J} = 0$. If $V \equiv 0$, then (3.1) becomes $d\tilde{V} = \tilde{V} df \circ J$, which is solved by $\tilde{V} = \ell_{\mathbf{u}} = \mathbf{u}\tilde{f}$, since it then reduces to (2.1). In the other direction, if $\tilde{V} \equiv 0$, then (3.2) reduces to $dV = 0$, giving $V = \tau_{\mathbf{u}}$.

We summarize the discussion above as follows:

Proposition 3.1. *The cousin operation $V \mapsto \tilde{V}$ on Jacobi fields, given by integrating (3.1), is an isomorphism from the space of Jacobi fields on M^+ modulo translations to the space of Jacobi fields on \tilde{M}^+ modulo left-translations. The inverse isomorphism is given by integrating (3.2). \square*

3.2. Examples. Trivial examples of Jacobi fields on M^+ and \tilde{M}^+ are the tangential fields and the restrictions of Killing fields on the respective ambient spaces, as described earlier. We can explicitly compute their cousins.

Lemma 3.2. *If V is a tangential vector field on M^+ , then $\tilde{V} = \tilde{f} J(V)$ is a cousin.*

Proof. Because V is tangential, we can pull it back under f , to get $V = df(X)$ for some tangent vector field X on Σ^+ . We think of X as the derivative of a one-parameter family of diffeomorphisms of Σ . Since the cousin equation (2.1) respects reparametrization in the sense that $\widetilde{f \circ \varphi} = \tilde{f} \circ \varphi$, it follows that $\tilde{V} = d\tilde{f}(X)$. Using (2.1) and the fact that df commutes with J , we get

$$\tilde{V} = \tilde{f} df(J(X)) = \tilde{f} J(df(X)) = \tilde{f} J(V). \quad \square$$

Here, we have avoided computing \dot{J} by not using (3.1) directly. When V is a Killing field, we know in advance that $\dot{J} = 0$. Of course, if $V = \tau_{\mathbf{u}} \equiv \mathbf{u}$ is a translation field, then $\tilde{V} \equiv 0$ is a cousin (and indeed the left-translations are the other cousins). It is more interesting to consider the rotational Killing fields.

Lemma 3.3. *The infinitesimal rotation $\rho_{\mathbf{u}}$ around axis \mathbf{u} has the right-translation $r_{\mathbf{u}}$ as a cousin.*

Proof. We want to show that $V = \rho_{\mathbf{u}} = f\mathbf{u} - \mathbf{u}f$ and $\tilde{V} = r_{\mathbf{u}} = \tilde{f}\mathbf{u}$ satisfy (3.1) with $\dot{J} = 0$. But

$$dV = df \mathbf{u} - \mathbf{u} df, \quad d\tilde{V} = d\tilde{f} \mathbf{u} = \tilde{f} (df \circ J) \mathbf{u},$$

so indeed $d\tilde{V} = \tilde{V} df \circ J + \tilde{f} dV \circ J$. \square

Example. Suppose $M = \mathbb{S}^2$ is a sphere, with $f = \tilde{f}$, and suppose $V = \rho_{\mathbf{u}} = -2\mathbf{u} \times f$ is a rotational Killing field. The symmetry of the sphere means that V is tangential, so by Lemma 3.2 one cousin is $\tilde{V} = 2\tilde{f}J(f \times \mathbf{u})$. Using the facts that for \mathbb{S}^2 we have $J(\mathbf{v}) = \mathbf{v} \times f$ and $f^2 = -1$, we can simplify this as follows:

$$\begin{aligned}\tilde{V} &= 2f((f \times \mathbf{u}) \times f) = f((f \times \mathbf{u})f - f(f \times \mathbf{u})) \\ &= \tfrac{1}{2}f(2f\mathbf{u}f + 2\mathbf{u}) = f\mathbf{u} - \mathbf{u}f = \tilde{f}\mathbf{u} - \mathbf{u}\tilde{f} = r_{\mathbf{u}} - \ell_{\mathbf{u}}.\end{aligned}$$

Of course, by Lemma 3.3, another cousin of $V = \rho_{\mathbf{u}}$ is $r_{\mathbf{u}}$. These two cousins are not equal, but their difference is of course a left-translation field, in this case $\ell_{\mathbf{u}}$.

3.3. Transplanting the cousin. Given an \mathbb{R}^3 -valued vector field V on M^+ , which we think of as a function $V: \Sigma^+ \rightarrow \mathbb{R}^3$, there is a *transplanted* vector field $\tilde{f}V$ on \tilde{M}^+ , whose value $\tilde{f}(x)V(x)$ at $\tilde{f}(x)$ (given by quaternion multiplication) is tangent to \mathbb{S}^3 there. That is, we use our identification of different tangent spaces to \mathbb{S}^3 via left-translation to give a natural isomorphism $V \leftrightarrow \tilde{f}V$ between the space of vector fields on M^+ and the space of such $T\mathbb{S}^3$ -valued fields on \tilde{M}^+ . We will most often transplant fields from \tilde{M}^+ to M^+ , so we introduce the following notation: Given a field W on \tilde{M}^+ , we write $\bar{W} := \tilde{f}^{-1}W$ for the transplant back to M^+ .

One can easily check that W is a Jacobi field for \tilde{M}^+ if and only if its transplant \bar{W} is a Jacobi field for M^+ . (See [KKR, Lemma 6].)

Independent of the particular surface M^+ , the transplant to M^+ of a right-translation field $r_{\mathbf{u}}$ on \tilde{M}^+ is the corresponding translational field: $\bar{r}_{\mathbf{u}} = \tau_{\mathbf{u}}$. On the other hand, transplantation of the other Killing fields gives nontrivial examples of Jacobi fields. Of particular interest is the transplant $\bar{\ell}_{\mathbf{u}}$, a Jacobi field on M^+ given by

$$\bar{\ell}_{\mathbf{u}}(\tilde{f}(x)) = \tilde{f}^{-1}(x)\mathbf{u}\tilde{f}(x) = \Pi_{\mathbf{u}}(\tilde{f}^{-1}(x)).$$

As discussed in [KKR, Appendix A], one can recover this Hopf projection of the conjugate cousin surface from spinning spheres, giving an alternative description of the CMC condition.

We now return to a general transplant $\bar{W} = \tilde{f}^{-1}W$. Using $d\tilde{f}^{-1} = -\tilde{f}^{-1}d\tilde{f}\tilde{f}^{-1}$ with (2.1), we get $d\bar{W} = d(\tilde{f}^{-1}W) = (d\tilde{f}^{-1})W + \tilde{f}^{-1}dW = -\tilde{f}^{-1}d\tilde{f}\tilde{f}^{-1}W + \tilde{f}^{-1}dW = -(df \circ J)\bar{W} + \tilde{f}^{-1}dW$, which gives

$$(3.3) \quad \tilde{f}^{-1}dW = d\bar{W} + (df \circ J)\bar{W}.$$

The derivative $d\bar{W}$ can also be interpreted as the covariant derivative of W with respect to the flat connection on \mathbb{S}^3 given by left-translation (while dW , on the other hand, is the derivative in \mathbb{H}).

As an example, if $W = \ell_{\mathbf{u}} = \mathbf{u}\tilde{f}$, then we get $d\ell_{\mathbf{u}} = \mathbf{u}d\tilde{f} = \ell_{\mathbf{u}}(df \circ J)$. Then by (3.3) we have

$$d\bar{\ell}_{\mathbf{u}} = \bar{\ell}_{\mathbf{u}}(df \circ J) - (df \circ J)\bar{\ell}_{\mathbf{u}} = 2\bar{\ell}_{\mathbf{u}} \times (df \circ J),$$

or equivalently

$$(3.4) \quad 2df \times \bar{\ell}_{\mathbf{u}} - d\bar{\ell}_{\mathbf{u}} \circ J = 0.$$

We note that the conjugation operation discussed extensively in [KKR] was the map $V \mapsto \widetilde{V}^\perp$, sending a Jacobi field V to the normal part of the transplant \widetilde{V} of its cousin \widetilde{V} . Let us consider this transplanted cousin \widetilde{V} . Using (3.3), we can rewrite (3.2) as

$$\begin{aligned} (3.5) \quad dV &= df \widetilde{V} - \widetilde{V} df - d\widetilde{V} \circ J + df \circ \dot{J} \circ J \\ &= 2 df \times \widetilde{V} - d\widetilde{V} \circ J + df \circ \dot{J} \circ J, \end{aligned}$$

where, in the second line, we have used the fact that both df and \widetilde{V} have values in $\mathbb{R}^3 = \text{Im } \mathbb{H}$ to rewrite the quaternionic commutator as a vector cross product. Note that when $V = \tau_{\mathbf{u}}$ is constant (so $dV = 0$ and $\dot{J} = 0$) then the conjugate is $\widetilde{V} = \bar{\ell}_{\mathbf{u}}$; in this case (3.5) reduces to (3.4).

3.4. Almost even and almost odd fields. Given a coplanar k -unduloid M , we want to examine vector fields V on the upper half M^+ in terms of their behavior on the boundary. Remember that ∂M^+ consists of k curves γ_i , each lying in the horizontal \mathbf{ij} -plane of symmetry. The conjugate cousin surface \widetilde{M}^+ in \mathbb{S}^3 also has k boundary curves $\widetilde{\gamma}_i$; these lie along Hopf circles.

In Section 2.6, we defined V to be *even* if, on the boundary, V is horizontal and its normal derivative $\partial_n V$ is vertical. We can interpret this more geometrically by flowing the surface in the direction V . We see that even fields are exactly those for which the following properties hold to first order:

- (a) each boundary curve remains in the fixed \mathbf{ij} -plane with M^+ meeting that plane perpendicularly; and
- (b) the family of conormal curves on M^+ , meeting that boundary perpendicularly, is preserved.

The notion of *odd* fields is easiest to interpret geometrically when transplanted to \widetilde{M}^+ . Suppose W is a vector field on \widetilde{M}^+ . We say W is *odd* if its transplant \overline{W} is odd, that is, if \overline{W} is vertical along the boundary and its normal derivative $\partial_n \overline{W}$ is horizontal. Equivalently, at a point $p \in \partial \widetilde{M}^+$, this means that W is parallel to the \mathbf{k} -Hopf circle through p (along which the boundary curve lies) and that $\partial_n W$ is perpendicular to that Hopf circle. The geometric interpretation is that odd fields are exactly those for which, when we flow \widetilde{M}^+ in the direction W , the following hold to first order:

- (a*) each boundary curve remains along its fixed \mathbf{k} -Hopf circle; and
- (b*) the family of conormal curves on \widetilde{M}^+ , meeting that boundary perpendicularly, is preserved.

To appreciate conditions (b) and (b*), think of them as saying that orthogonality is preserved; this will later allow us to control the \dot{J} term in the cousin equations (3.1) and (3.2).

As mentioned in Section 2, our goal is to use cousin Jacobi fields to convert even fields (which we want to understand) to odd fields. Odd fields are already better understood thanks to the following result [KKR, Prop. 24]:

Lemma 3.4. *Any bounded odd Jacobi field $V \in \widehat{\mathcal{J}}(M)$ is (up to tangential components) a multiple of the vertical translation $\tau_{\mathbf{k}}$. In particular, there are no odd L^2 Jacobi fields.* \square

Unfortunately, the conjugate of an even field is in general not odd, but instead satisfies a slightly weaker condition, which we define now, motivated by the geometric interpretations above.

Definition. A field V on M^+ is *almost even* if along each boundary curve γ_i it differs from an even field by some translation $\tau_{\mathbf{v}_i}$; here we can take \mathbf{v}_i to be vertical. A field W on \tilde{M}^+ is *almost odd* if along each boundary curve $\tilde{\gamma}_i$ it differs from an odd field by some left-translation $\ell_{\mathbf{w}_i}$; here we can take \mathbf{w}_i perpendicular to $\Pi_{\mathbf{k}}(\tilde{\gamma}_i) \in \mathbb{S}^2$.

Whereas an even field (to first order) keeps each boundary curve γ_i in the \mathbf{ij} -plane, an almost even field translates each γ_i to a parallel plane, with velocity \mathbf{v}_i . Similarly, whereas an odd field preserves (to first order) each Hopf circle $\tilde{\gamma}_i$, an almost odd field will left-translate it to another Hopf circle. To quantify this, we compute the derivative of Hopf projection.

Lemma 3.5. *Left-translation Hopf-projects to rotation: For a unit imaginary $\mathbf{u} \in \mathbb{S}^2$ and any $p \in \mathbb{S}^3$, we have $\Pi_{\mathbf{k}}(e^{t\mathbf{u}}p) = e^{t\mathbf{u}}\Pi_{\mathbf{k}}(p)e^{-t\mathbf{u}}$. Infinitesimally,*

$$d_p \Pi_{\mathbf{k}}(\ell_{\mathbf{u}}(p)) = d_p \Pi_{\mathbf{k}}(\mathbf{u}p) = 2\mathbf{u} \times \Pi_{\mathbf{k}}(p) = -\rho_{\mathbf{u}}(\Pi_{\mathbf{k}}p).$$

That is, $d\Pi_{\mathbf{k}}(\ell_{\mathbf{u}}) = -\rho_{\mathbf{u}}$, in the sense that the vector field $\ell_{\mathbf{u}}$ is $\Pi_{\mathbf{k}}$ -related to $-\rho_{\mathbf{u}}$.

Proof. The definition $\Pi_{\mathbf{k}}(p) := pkp^{-1}$ immediately yields the equation for $\Pi_{\mathbf{k}}(e^{t\mathbf{u}}p)$. Differentiating this at $t = 0$ gives

$$d_p \Pi_{\mathbf{k}}(\mathbf{u}p) = \left. \frac{d}{dt} \right|_{t=0} e^{t\mathbf{u}} \Pi_{\mathbf{k}}(p) e^{-t\mathbf{u}} = \mathbf{u} \Pi_{\mathbf{k}}(p) - \Pi_{\mathbf{k}}(p) \mathbf{u} = 2\mathbf{u} \times \Pi_{\mathbf{k}}(p). \quad \square$$

We note that, according to this formula, the only two \mathbf{k} -Hopf circles remaining fixed under $\ell_{\mathbf{u}}$ are the ones projecting to $\pm \mathbf{u} \in \mathbb{S}^2$. This makes sense, since along these circles, $\ell_{\mathbf{u}} = \mathbf{u}p = pk = r_{\mathbf{k}}$ is tangent.

Lemma 3.6. *The almost odd fields on \tilde{M}^+ are exactly those which have a well-defined first-order effect on*

$$\Pi_{\mathbf{k}}(\partial \tilde{M}^+) = (\mathbf{p}_1, \dots, \mathbf{p}_k) \in (\mathbb{S}^2)^k;$$

this is given by $\rho_{\mathbf{w}_i}$ at \mathbf{p}_i .

Proof. An odd field preserves the boundary Hopf circles by property (a*) above. An almost odd field W differs from an odd field by $\ell_{\mathbf{w}_i}$ along $\tilde{\gamma}_i$. Thus by Lemma 3.5 its action on $\mathbf{p}_i := \Pi_{\mathbf{k}}(\tilde{\gamma}_i)$ is the rotation $\rho_{\mathbf{w}_i}$. (Note that the action of $\rho_{\mathbf{w}_i}$ on \mathbf{p}_i is well-defined, since different choices of \mathbf{w}_i differ by multiples of \mathbf{p}_i .) \square

For local computations along the boundary curve γ_i , we pick an orthonormal coordinate frame (∂_t, ∂_n) , where $\partial_n = J\partial_t$ and $\tau := \partial_t f = -\tilde{f}^{-1}\partial_n \tilde{f}$ is the horizontal tangent vector along γ_i , while $\mathbf{k} = \partial_n f = \tilde{f}^{-1}\partial_t \tilde{f}$ is the constant vertical conormal. In these coordinates, we can give an alternate characterization of almost even and almost odd fields.

Along a boundary curve, an almost even field differs from an even one by a constant. Thus it is clear that a field V on M^+ is almost even if and only if, along the boundary ∂M^+ , the tangential derivative $\partial_t V$ is horizontal, while the normal derivative $\partial_n V$ is vertical. We think of these conditions as properties of the vector-valued 1-form dV , and have thus proved the first half of the following lemma:

Lemma 3.7. *A vector field V on M^+ is almost even if and only if, along the boundary ∂M^+ , the one form dV is horizontal when applied to ∂_t and vertical when applied to ∂_n . Similarly, a vector field W on \tilde{M}^+ is almost odd if and only if the form $2df \times \bar{W} - d\bar{W} \circ J$ is horizontal when applied to ∂_t and vertical when applied to ∂_n .*

Proof. To prove the second statement, let $\omega := 2df \times \bar{W} - d\bar{W} \circ J$. By (3.4), this form ω is unchanged if we add a left-translation ℓ_u to W .

Therefore, given an almost odd W , we are free to assume it is odd along any given boundary curve, that is, that \bar{W} is vertical and $\partial_n \bar{W}$ is horizontal. Then $\omega(\partial_t) = 2\tau \times \bar{W} - \partial_n \bar{W}$ is indeed horizontal, and $\omega(\partial_n) = 2\mathbf{k} \times \bar{W} + \partial_t \bar{W} = \partial_t \bar{W}$ is indeed vertical.

Conversely, given W for which ω has the given properties, we are free to assume that \bar{W} is vertical at some initial boundary point. At any point where \bar{W} is vertical, $\mathbf{k} \times \bar{W}$ vanishes, so $\partial_t \bar{W} = \omega(\partial_n)$ is also vertical. Thus, in the unique solution to the ODE, we find that \bar{W} stays vertical along the whole boundary curve. \square

Proposition 3.8. *Suppose V is a Jacobi field on M^+ and \tilde{V} is a cousin field on \tilde{M}^+ . Then V is almost even if and only if \tilde{V} is almost odd.*

Proof. Rewriting equation (3.5) as

$$dV - (2df \times \tilde{V} - d\tilde{V} \circ J) = df \circ \dot{J} \circ J,$$

we see by Lemma 3.7 that it suffices to prove that $df \circ \dot{J} \circ J$ is horizontal when applied to ∂_t and vertical when applied to ∂_n .

When flowing by an even field V or an odd field \tilde{V} , it follows from properties (a,b) or (a*,b*) above that, to first order, the frame (∂_t, ∂_n) remains orthogonal along γ_i . The same is true if we flow by an almost even or almost odd field, since locally the only difference is a Killing field.

Thus in this frame, to first order, J flows from $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to a matrix of the form $\begin{pmatrix} 0 & -1/a \\ a & 0 \end{pmatrix}$. That is, $\dot{J} = \begin{pmatrix} 0 & \dot{a} \\ \dot{a} & 0 \end{pmatrix}$, implying that $\dot{J} \circ J$ is diagonal. Thus $df \circ \dot{J} \circ J(\partial_t)$ is horizontal, a multiple of τ , while $df \circ \dot{J} \circ J(\partial_n)$ is vertical, a multiple of \mathbf{k} . \square

3.5. The differential of the classifying map. We now have all the ingredients we need to compute the formal differential of the classifying map $\Phi: \mathcal{M}'_k \rightarrow \mathcal{D}_k$. Recall that Φ takes M to the Hopf projection of the conjugate cousin \tilde{M}^+ of its upper half. We write $(\mathbf{p}_1, \dots, \mathbf{p}_k) \in \mathcal{T}_k$ for the boundary of $\Phi(M)$.

Given an almost even field V on M^+ , by Proposition 3.8 any conjugate \tilde{V} is almost odd, differing from an odd field by some $\ell_{\mathbf{w}_i}$ on each $\tilde{\gamma}_i$. Thus, by Lemma 3.6, \tilde{V} has the well-defined effect $-\rho_{\mathbf{w}_i}$ on each \mathbf{p}_i . Of course, \tilde{V} itself is only well-defined up to some left-translation ℓ_u , but this just gives a global rotation $-\rho_u \in \mathfrak{so}_3$ of the whole k -tuple, so we end up with a well-defined element of $T_{\Phi(M)}\mathcal{D}_k$, using the characterization (2.2) of that space.

Proposition 3.9. *The formal differential $\partial\Phi: \mathcal{J}'(M) \rightarrow T_{\Phi(M)}\mathcal{D}_k$, given by*

$$\partial\Phi([V]) := (-\rho_{\mathbf{w}_1}, \dots, -\rho_{\mathbf{w}_k}),$$

is well-defined. Moreover, when V is integrable (i.e., when $[V] \in T_M \mathcal{M}'_k$) this is—as suggested by the notation—the derivative of the classifying map Φ .

Proof. Suppose $[V] \in \mathcal{J}'(M)$ is given, with V an even Jacobi field. The construction of $\partial\Phi$ parallels that of Φ , making the last statement clear by the chain rule.

The only thing left to check is that $\partial\Phi([V])$ is independent of the even representative $V \in [V]$. But this is straightforward: An even Killing field is either a horizontal translation (with no effect on \tilde{V}) or the rotation ρ_k ; by Lemma 3.3, the latter has as a cousin r_k , which is tangent to \mathbf{k} -Hopf circles and thus has no effect on the \mathbf{p}_i . Similarly, an even tangential field is horizontal on ∂M^+ , so by Lemma 3.2 it has a tangential cousin which is vertical (that is, again tangent to \mathbf{k} -Hopf circles) on the boundary, so again has no effect. \square

We can easily give a preliminary characterization of the kernel of this map; this lemma will later be used in showing that in fact the kernel vanishes.

Lemma 3.10. *For $[V] \in \mathcal{J}'(M)$, we have $\partial\Phi([V]) = 0$ if and only if V has an odd cousin \tilde{V} .*

Proof. Any cousin \tilde{V} is almost odd, meaning that it differs from an odd field by some $\ell_{\mathbf{w}_i}$ along $\tilde{\gamma}_i$. Thus \tilde{V} is odd if and only if the vectors \mathbf{w}_i can be chosen to vanish. Of course, the definition of $T\mathcal{D}_k$ means that $\partial\Phi([V]) = 0$ even if the k -tuple is rotating, that is, if all the \mathbf{w}_i can be chosen equal to some fixed \mathbf{u} . But then $\tilde{V} - \ell_{\mathbf{u}}$ is another cousin for which $\mathbf{w}_i = 0$; it is thus odd. \square

4. MODULI SPACE THEORY AND THE DIFFERENTIAL OF THE ASYMPTOTES MAP

In this section, we review some basic structure results about CMC moduli spaces. (See [KMP] and the analogous discussion of constant scalar curvature metrics in [MPU].) Then we use the linear decomposition lemma to compute the (formal) differential of the asymptotes map $\mathcal{A} : \hat{\mathcal{M}}_{g,k} \rightarrow \mathcal{U}^k$.

4.1. Delaunay unduloids. Recall that an unduloid has a conformal parametrization of the form

$$U : (t, \theta) \mapsto x(t) \mathbf{i} + r(t) e^{\theta \mathbf{i}} \mathbf{j}.$$

Here U is an unduloid with necksize n , positioned so that its axis is the \mathbf{i} -axis and one neck lies in the \mathbf{jk} -plane. Thus $x(0) = 0$ and r assumes its minimum of $n/2\pi$ at $t = 0$.

We let η be the Jacobi field arising from differentiation with respect to the necksize n ; we call η the necksize-change field. (On the cylinder, where $n = \pi$, the space of necksize-change fields is two-dimensional, since necks can be inserted at any phase along the axis.)

The necksize-change and Killing fields together form a six-dimensional subspace of $\hat{\mathcal{J}}(U)$. (Note that ρ_i is tangential, so does not contribute here.) The translation fields are bounded (in fact, periodic in t), while η and the rotation fields grow linearly t . (On the cylinder the translation τ_i is also tangential, but the extra necksize change compensates in the dimension count. Here, the necksize changes are bounded.)

Using the rotational symmetry of U to separate variables and expand any Jacobi field in a Fourier series, we see that the modes of order 0 and ± 1 correspond to the geometric motions described above, while the higher Fourier modes grow exponentially on at least one end ($t \rightarrow \infty$ or $t \rightarrow -\infty$).

On the other hand, if $u \in L^2([0, \infty) \times \mathbb{S}^1)$ is a Jacobi field, then u must decay exponentially, and there is a lower bound (independent of the necksize n) for the rate of exponential decay.

We can summarize this discussion as follows. (See also [KKS], [KMP], and [MP, Prop. 20].)

Lemma 4.1. *The unduloid U with necksize n is nondegenerate. Indeed, the space $\widehat{\mathcal{J}}(U)$ of tempered Jacobi fields is six-dimensional, spanned by the necksize-change and Killing fields. These fields all grow at most linearly in t , and are all integrable, meaning $\widehat{\mathcal{J}}(U)$ is the tangent cone $T_U \mathcal{U}$ to the premoduli space $\mathcal{U} = \widehat{\mathcal{M}}_{0,2}$. The even part $\widehat{\mathcal{J}}'(U) = T_U \mathcal{U}'$ is four-dimensional, spanned by the necksize-change, the horizontal translations $(\tau_{\mathbf{i}}, \tau_{\mathbf{j}})$, and the rotation $\rho_{\mathbf{k}}$. \square*

4.2. The Jacobi operator on finite-topology CMC surfaces. Fixing a CMC surface $M \in \widehat{\mathcal{M}}_{g,k}$, we want to understand the asymptotic behavior of tempered Jacobi fields on M . Following [KMP, p. 126], choose a representative E_i for each end that is a normal graph over one end of an unduloid U_i ; this allows us to identify E_i with a subset of U_i . Cover M with the E_i plus a compact set E_0 and let $\{\varphi_i : 0 \leq i \leq k\}$ be a partition of unity subordinate to this cover. We define the *deficiency space*

$$\mathcal{W} = \mathcal{W}(M) := \left\{ W = \sum_{i=1}^k \varphi_i W_i : W_i \in \widehat{\mathcal{J}}(U_i) \right\},$$

and for a coplanar $M \in \widehat{\mathcal{M}}'_{g,k}$, we further define the *even deficiency subspace*

$$\mathcal{W}' := \left\{ W = \sum_{i=1}^k \varphi_i W_i : W_i \in \widehat{\mathcal{J}}'(U_i) \right\},$$

assuming we have chosen the partition of unity symmetrically.

The elements $W \in \mathcal{W}$ are not Jacobi fields, but their normal parts $\langle W, \nu \rangle$ have the property that, along each end, $\mathcal{L}(\langle W, \nu \rangle)$ decays exponentially in the coordinate t , at a rate depending only on the asymptotic necksize of that end. One may also regard \mathcal{W} as a quotient space of functions on M which are asymptotic to some tempered Jacobi field on each end, modulo functions asymptotic to zero; we use this interpretation implicitly in the Remark below Proposition 4.5.

To distinguish tempered Jacobi fields from those which grow exponentially on at least one end, we use weighted Sobolev spaces $H_\delta^s(M)$. We say $u \in H_\delta^s$ if $u \in H_{\text{loc}}^s$ and, when restricted to any end, $e^{-\delta t} u \in H^s$. We write the Jacobi operator \mathcal{L} as

$$\mathcal{L}_\delta : H_\delta^{s+2} \rightarrow H_\delta^s$$

to emphasize the choice of weight δ for the domain and codomain.

We recall the linear decomposition lemma [KMP, Lemma 2.9]:

Lemma 4.2. *There exists $\gamma > 0$, depending only on the necksizes of M , such that for any $\delta \in (0, \gamma)$ any function $u \in H_\delta^{s+2}(M)$ with $\mathcal{L}(u) \in H_{-\delta}^s(M)$ can be decomposed as $u = \langle W, \nu \rangle + v$, where $W \in \mathcal{W}$ and $v \in H_{-\delta}^{s+2}(M)$; in particular, u has at most linear growth along each end. \square*

4.3. The differential of the asymptotes map. An immediate corollary of the linear decomposition lemma is the existence of a formal differential $\partial\mathcal{A}$ for the asymptotes map:

Corollary 4.3. *The asymptotes map $\mathcal{A}: \widehat{\mathcal{M}}_{g,k} \rightarrow \mathcal{U}^k$ has a formal differential $\partial\mathcal{A}$ which is defined for all tempered Jacobi fields $V \in \widehat{\mathcal{J}}(M)$, and which coincides with the derivative of \mathcal{A} on integrable fields.*

Proof. Given $V \in \widehat{\mathcal{J}}(M)$, Lemma 4.2 gives $V = \sum \varphi_i W_i + v$, with v exponentially decaying. This means V is asymptotic to $W_i \in \widehat{\mathcal{J}}(U_i)$ on end E_i . We now set $\partial\mathcal{A}(V) := (W_1, \dots, W_k)$. When V is integrable, the W_i must agree with the actual first-order changes of the asymptotes, so this $\partial\mathcal{A}$ is indeed the differential of \mathcal{A} . \square

Given a Jacobi field $V \in \widehat{\mathcal{J}}(M)$, we now have two ways of measuring the rate of change it induces in the asymptotic necksize n_i : we can compute $\partial\Phi([V])$ and find the rate of change of distance from \mathbf{p}_{i-1} to \mathbf{p}_i (Proposition 3.9), or we can compute $\partial\mathcal{A}(V)$ and look at the necksize-change component of W_i (Corollary 4.3). Clearly the computations agree for integrable fields, since they both measure the actual change of necksize along any one-parameter family of CMC surfaces. In fact, they always agree:

Lemma 4.4. *The rate of change of necksize n_i under $V \in \widehat{\mathcal{J}}(M)$ can be computed equivalently from $\partial\Phi$ or from $\partial\mathcal{A}$.*

Proof. Since V and W_i are exponentially asymptotic along E_i , they have the same effect on \mathbf{p}_{i-1} and \mathbf{p}_i . But W_i is an integrable field on an unduloid U_i , so its necksize-change component agrees with the rate of change of distance from \mathbf{p}_{i-1} to \mathbf{p}_i . \square

4.4. Dimension counting with (coplanar) symmetry. The linear decomposition lemma applies in particular to tempered Jacobi fields, showing their normal parts have at most linear growth. Since no nonzero element of \mathcal{W} has normal part in L^2 , we also see that L^2 normal Jacobi fields must decay exponentially on each end. More precisely, if we write $\widehat{\mathcal{J}}_0(M)$ for the space of L^2 Jacobi fields and $\widehat{\mathcal{J}}'_0(M)$ for the subspace of even fields, then for sufficiently small $\delta > 0$, we find that $\widehat{\mathcal{J}}(M) \cong \ker(\mathcal{L}_\delta)$ and similarly $\widehat{\mathcal{J}}_0(M) \cong \ker(\mathcal{L}_{-\delta})$.

The duality between \mathcal{L}_δ and $\mathcal{L}_{-\delta}$ suggests that we use the relative index theorem of Melrose [Mel, Sec. 6.2] to compute the difference in dimension of $\widehat{\mathcal{J}}(M)$ and $\widehat{\mathcal{J}}_0(M)$. Indeed, if $M \in \mathcal{M}_{g,k}$ has any finite isometry group $G < \mathrm{SO}_3$ (acting perhaps to permute the ends), we can consider the spaces of G -invariant Jacobi fields, and we get:

Proposition 4.5. *For any G -symmetric $M \in \mathcal{M}_{g,k}$ we have*

$$\dim \widehat{\mathcal{J}}^G(M) - \dim \widehat{\mathcal{J}}_0^G(M) = \frac{1}{2} \dim \mathcal{W}^G.$$

Here, $\dim \mathcal{W}^G$ can be computed as a sum over a set of representative ends, inequivalent under G . The contribution of each end is either 6 (for an end in general position), or 4 (for an end in a mirror plane) or 2 (for an end along a rotation axis of G). The first two cases are already familiar to us from Lemma 4.1; in the last case, of course, the only symmetric perturbations of the end are necksize change and translation along the axis.

The case of no symmetry, $G = 1$, was proved as [KMP, Thm. 2.11]. The general case follows from the same relative index calculation, simply by restricting throughout to symmetric functions. (Specific cases have been used, for instance, in [JP, Sect. 4.3] and [HP, Props. 2.1, 2.2].) For the case of coplanar surfaces, $G \cong \mathbb{Z}_2$ is mirror symmetry, and the result becomes

$$(4.1) \quad \dim \widehat{\mathcal{J}}'(M) - \dim \widehat{\mathcal{J}}'_0(M) = \frac{1}{2} \dim \mathcal{W}' = 2k.$$

Remark. A more conceptual proof of Proposition 4.5 begins by observing that \mathcal{W} is a symplectic vector space with respect to the Gauss–Green form [KMP], and that the bounded nullspace $\mathcal{B} = \mathcal{B}(M) := \widehat{\mathcal{J}}(M)/\widehat{\mathcal{J}}_0(M)$ may be regarded as an isotropic subspace of \mathcal{W} . The Melrose relative index theorem implies $\dim \mathcal{B} = 3k = \frac{1}{2} \dim \mathcal{W}$ (whether or not M is nondegenerate), so \mathcal{B} is Lagrangian in \mathcal{W} . If G acts on M by isometries, it induces a symplectic G -action on \mathcal{W} preserving \mathcal{B} . But it follows from a symplectic linear algebra lemma [GKS2, Lemma 6.2] that the fixed-point set \mathcal{W}^G is a symplectic subspace of \mathcal{W} , and also that $\widehat{\mathcal{J}}^G(M)/\widehat{\mathcal{J}}_0^G(M) = \mathcal{B}^G = \mathcal{B} \cap \mathcal{W}^G$ is Lagrangian in \mathcal{W}^G , yielding the result.

The following result ([KMP, Thm. 3.1], as reinterpreted in [Kus]) explains the connection between nondegeneracy of M and the regularity of the asymptotes map near M .

Proposition 4.6. *If M is nondegenerate then there exists a neighborhood of M in $\widehat{\mathcal{M}}_{g,k}$ which is a manifold of dimension $3k$, and on which the asymptotes map $\mathcal{A} : \widehat{\mathcal{M}}_{g,k} \rightarrow \mathcal{U}^k$ is an embedding. \square*

The analagous statement for coplanar surfaces appears as [GKS2, Thm. 5.2]. Combining it with equation (4.1), we get the following:

Theorem 4.7. *For a coplanar $M \in \mathcal{M}'_{g,k}$, we have $\dim \widehat{\mathcal{J}}'(M) \geq 2k$, with equality if and only if M is nondegenerate. Also, if M is nondegenerate then, in a neighborhood of M , the premoduli space $\widehat{\mathcal{M}}'_{g,k}$ is a real-analytic $2k$ -manifold with tangent space $\widehat{\mathcal{J}}'(M)$, and the moduli space $\mathcal{M}'_{g,k}$ is a $(2k - 3)$ -manifold with tangent space $\mathcal{J}'(M)$. \square*

5. PROOFS OF THE MAIN THEOREMS

The key to our main results is the following theorem, guaranteeing the injectivity of $\partial\Phi$.

Theorem 5.1. *The formal differential $\partial\Phi : \mathcal{J}'(M) \rightarrow T_{\Phi(M)}\mathcal{D}_k$ of the classifying map Φ , as given in Proposition 3.9, is injective. Thus $\dim \mathcal{J}'(M) \leq 2k - 3$, so $\dim \widehat{\mathcal{J}}'(M) \leq 2k$.*

Proof. Suppose $V \in \widehat{\mathcal{J}}'(M)$ is an even field with $\partial\Phi([V]) = 0$. By Lemma 3.10, it has an odd cousin \widetilde{V} . We claim the normal part of \widetilde{V} is bounded. Then by Lemma 3.4 its transplant is a vertical translation τ_{ak} . Equivalently, $\widetilde{V} = r_{ak}$. (All our computations are up to tangential components.) But we know that r_{ak} is a cousin of ρ_{ak} , so it follows that $V = \rho_{ak} + \tau_{\mathbf{u}}$ for some $\mathbf{u} \in \mathbb{R}^3$. But this is a Killing field, meaning that $[V] = 0 \in \mathcal{J}'(M)$. The dimension bounds follow immediately from the known dimension of \mathcal{D}_k .

To prove the claim, we look at $\partial\mathcal{A}(V) = (W_1, \dots, W_k)$. Here $W_i \in \widehat{\mathcal{J}}'(U_i)$ where U_i is the asymptote of end E_i . Since $\partial\Phi([V]) = 0$, by Lemma 4.4, the fields W_i include no necksize changes. Thus each W_i is an even Killing field, in the span of $\{\tau_i, \tau_j, \rho_k\}$. But we know cousins for the Killing fields: translations have vanishing cousins, while ρ_k has cousin r_k by Lemma 3.3. Thus

W_i has a bounded cousin r_{ak} , so V has a cousin which along E_i is exponentially asymptotic to this right-translation, and in particular is bounded. Of course, since left-translations are bounded, the boundedness of some cousin along each end implies the boundedness of any cousin along all ends; in particular, the odd cousin \tilde{V} is bounded. \square

Nondegeneracy Theorem. *All coplanar k -unduloids are nondegenerate.*

Proof. Combining Theorems 4.7 and 5.1, we get $\dim \hat{\mathcal{J}}'(M) = 2k$; the equality in Theorem 4.7 then implies nondegeneracy. \square

Diffeomorphism Theorem. *The classifying map $\Phi: \mathcal{M}'_k \rightarrow \mathcal{D}_k$ of [GKS2] is a real-analytic diffeomorphism.*

Proof. Using the Nondegeneracy Theorem and Theorem 4.7, we see that \mathcal{M}'_k is a real-analytic manifold of dimension $2k - 3$, with tangent space $\mathcal{J}'(M)$. The classifying map Φ is thus a real-analytic map between manifolds of the same dimension, and Theorem 5.1 says its differential is injective. The theorem then follows by the real-analytic inverse function theorem. \square

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